THE
MATHEMATICAL GAZETTE.

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OBITUARY.

THE REVEREND GEORGE RICHARDSON, who died at his son's residence in London on January 16th in his 64th year, will be missed by a large circle of friends. Born at Carlisle in 1840, he was educated at Chester, and on leaving school he obtained a Scholarship at St. John's College, Cambridge, at the age of 17. He graduated as Third Wrangler in 1860, the year in which Lord Justice Stirling was Senior Wrangler. In due course he was elected a Fellow of his College, and in 1867 he was appointed an Assistant Master at Winchester College by Dr. Ridding. From 1873 to 1899 he was Second Master at Winchester with charge of the scholars of the College.

Whilst at Cambridge he is said to have been able to read 8 or 10 hours a day without feeling any ill effects; and he carried the same amount of energy into his work at Winchester. His pupils felt that he had the art of putting things clearly, and a keen insight into the sources of their difficulties; and, echoing Goethe's words, they might have said: "Mit dem Genie haben wir am liebsten zu thun." Thoroughly successful in all that he undertook, he had a firm grip on the practical side of life; and, as sometimes happens, he was perhaps a little out of sympathy with the unsuccessful idealist. He told the present writer that he had "once spent several months in the study of metaphysics, and thought that there was very little in the subject." Though somewhat reserved, he was essentially "gemütlich," *i.e.* he had the comfortable and kindly good nature which made his pupils feel at their ease. It was, in fact, just this genial power and apparent absence of effort, which seemed so inspiring. The high places afterwards attained by many of his pupils in the Mathematical Tripos testify to the excellence of his teaching.

He felt the difficulties inseparable from the present public school system; and, with wide outlook on human nature, he thought there must be some way of surmounting them. On being presented with his portrait at Winchester in 1899, he strongly advocated the importance of educating boys and girls together. Perhaps the next generation will see the wisdom of the efforts already being made in the direction of co-education, and will endeavour to extend it.

Mr. Richardson was an original member of the Mathematical Association, founded as the Association for the Improvement of Geometrical Teaching in January, 1871, and in 1883 he was elected a Vice-President. At the annual meetings he took a leading part in the discussions, which are full of useful hints

on the art of teaching Mathematics (see the Annual Reports, 1-19, some of which are still in print). In 1873 he wrote a book on Geometrical Conics. In 1887 he read a paper on "The Teaching of Modern Geometry," which was followed in 1894 by the publication of his "Modern Plane Geometry," written in conjunction with Mr. Ramsey on the lines of the Association's "Syllabus of Modern Plane Geometry." In *Gazette*, No. 32, there appeared a valuable paper on "The Trigonometry of the Tetrahedron."

Amongst his numerous friends was the late Mr. Wolstenholme, with whom he corresponded from time to time on mathematical subjects. He was also a life-long friend of Sir Robert Ball from the time when they were at School at Chester together. With his friend the Rev. W. D. Bushell (late Fellow and Tutor of St. John's Coll. and afterwards Mathematical Master at Harrow), he took a keen interest in the Cambridge Rifle Corps; it is not perhaps generally known that Mr. Richardson was once second for the Queen's Prize.

Mr. Richardson had great powers of organisation; and if he had been a classic instead of a mathematician, he would probably have been appointed Headmaster at one of the great public schools. His genial and exceptional powers adorned his profession, whilst at the same time he was

"Rich in saving common-sense,
And, as the greatest only are,
In his simplicity sublime."

H. D. ELLIS.

The following extracts from a letter to the Editor from Sir Robert S. Ball, F.R.S., need no comment:

"We were schoolfellows at Tarvin Hall, near Chester. He was there distinguished for his mathematical powers as a boy, and his excellent qualities of head and heart were well appreciated by boys and masters alike. The same school was afterwards transferred to Abbotts Grange at Chester. He was two or three classes above me, and that meant of course a wide gulf in those days. But I remember well how kindly and clearly he explained to me the phases of the moon, some fifty years ago, during a walk round the playground. In later life I saw but little of him. . . . I am glad to think we always preserved the old friendship, and even decades in which we saw nothing of each other did not prevent the pleasant revival when we did meet. I was much touched by seeing in the papers in his obituary notice that I was considered a lifelong friend of his. That I certainly was for half a century and more; but at the same time I could not but reproach myself for not having striven more during all those long years to know more of one for whom I had the warmest esteem and regard."

THE SCIENTIFIC RESULTS AND AIMS OF MODERN APPLIED MECHANICS.*

By PROFESSOR A. SOMMERFELD of Aachen. (Translated by R. M. MILNE.)

IT is very far from my purpose to construct an antithesis between a purely scientific and a technical conception of the Problem of Mechanics. The conflict which undoubtedly existed some years ago, and at times was waged with bitterness, has come to a satisfactory conclusion; and a generous appreciation of the two different directions of investigation has taken the place of unseemly strife. From my personal knowledge of the Technical

* Professor Sommerfeld has kindly given permission to publish this abridged translation of the address, which he delivered to the Naturwissenschaftliche Hauptgruppe in Sept. 1903. The complete address is to be found in the *Physikalische Zeitschrift*, Vol. IV. No. 26*b*, pp. 773-782.

I am indebted to my colleague Mr. C. S. Jackson for suggesting the translation and for occasional advice on the matter of abridgment, and on the rendering of some passages. The whole address is worthy of the attention of any teacher of elementary mechanics, who wishes to get some idea of the more advanced methods and the recent developments of mechanical teaching in Germany.

High School in Aachen, I at least can lay stress on the fact that I have been encouraged in the most ready manner by my technical colleagues of all departments.

If I were required to characterize the modern developments of Applied Mechanics, I might on the one hand point to the all-pervading demand for securing the experimental foundations of the science or on the other hand to the application of the more powerful theoretical methods.

One cannot be surprised if in many branches of Applied Mechanics the experimental foundation is insecure, treated as they are at the outset by traditional rules, the application of which in the special case gives pause for thought. In the first place, experimental enquiry and criticism reduces every Natural Science to what it really ought to be, an Investigation of Nature. The erection of the new research laboratories in our High Schools has afforded this opportunity to the budding engineer. The engineers of machine construction took the lead in the demand for experimental research. They have to-day the satisfaction of possessing richly equipped laboratories for research and instruction in almost all German High Schools. The requirements of structural engineering soon gave rise to a corresponding demand. It is hoped that it will not be long before every High School has also its laboratory for the study of the problems presented by the employment of stone and iron in structural work. In former times only few students, by reason of their specially fortunate position, were able to carry out regular experiments in the sphere of Applied Mechanics. To-day however every student has at least for six months the opportunity of seeing with his own eyes the scientific problems of the subject in experimental form.

I might allude to the Theory of Earth Pressure as one branch, in which experimental research was urgently required. This theory, which we have received from the hands of Coulomb, Poncelet and Rankine, were it ever so plausible, can scarcely be maintained *a priori*. It transfers the laws of sliding friction, which hold good for solid dry bodies with smoothed surfaces, to the circumstances of earth with its ill-defined constitution, and works with the angle of friction for the sliding of earth on earth, or of earth on walls, without adducing proof that a real meaning attaches to these ideas for the case in hand. Obviously such a proof can only be obtained by experimental research: this has been undertaken many times without any conclusive result. It is gratifying that in the new laboratory for structural design in Charlottenburg, under the direction of Müller-Breslau, this problem has been given the first place. The results of the experiments, which have been planned on an extensive scale, are not completely published; we venture to hope that they will establish a firm foundation for this difficult subject.

Similar difficulties arise in all those branches of Mechanics in which friction plays an essential part. The mathematician and physicist are only too ready to avoid problems involving friction. But for the engineer the problem of friction is the problem of his life.

In my opinion the teacher of Mechanics should bear in mind that the original source of all natural knowledge is experiment. No one in these days dreams of introducing the beginner in Physics and Chemistry to the subject without confirming his lessons by extensive experiments. Why should not Mechanics also travel along the "royal road of experiment"? The older books on mechanics are of a strong deductive and almost dogmatic character. The learner might readily get the impression that the rigid structure of laws and proofs was complete and unbroken like Euclid's Elements, as though all that was required of experiment was that it should occasionally provide a numerical constant. I do not believe that this is the spirit of modern science. Rather am I of opinion that it is quite as instructive to be directed to the lack of theory as to be compelled to admire its pretended completeness. The time that is spent in instruction by way of experiments in Mechanics is richly

repaid by a deepening and quickening of conception, whereby a memory picture of definite measurements and force magnitudes is added to the abstract mathematical laws. In particular I have made successful experiments with the beautiful apparatus of Töpler,* which permits of the verification of all the fundamental laws on the equilibrium and the motion of bodies.

According to inquiries I have made, the Government is most ready to afford the means for putting life into instruction in Mechanics. In view of measures, which have been taken at other High Schools also, with respect to apparatus for instruction, I am of opinion that the time will soon be past when Mechanics is presented to the learner in the mathematical garb of a mere discipline in calculation and graphical work. It is to the corresponding change which in the case of Chemistry and Physics took place some fifteen years ago that these sciences owe their present vitality.

One distinctive feature of the discussion, somewhat superficial to be sure, appears when we attempt to separate the interests of the structural engineer on the one hand and the machine engineer on the other. In the earlier times mechanical theory pre-eminently served the aims of the former. For this reason Statics and Graphics constitute the foundation of Applied Mechanics. The development of engine construction and Electro-technics has created a change in this respect. Dynamics presses forward more and more into the field of the engineer. Probably one might aver that the structural engineer employs Mechanics more extensively and the machine engineer with greater concentration. The total amount of computation and graphical work, which the construction of a bridge or roof demands, is doubtless greater in extent than the necessary mechanical principles for the design of a machine. Nevertheless in the latter lie the more abstruse problems. The elastic strains in the pieces of a machine are more various and in general more speculative (kühner) than those of the pieces of a structure; besides, it is here first of all that the complete system of Mechanics, that is to say Dynamics, comes into its inheritance.

Among the theoretical methods of structural engineering the principle of Work-variation takes a leading position. Just as the idea of work, by reason of the Principle of Virtual Work, is all-powerful in Statics, so it gives their simplest form to the Statics of structures under elastic strains, as soon as the expression for the work of the elastic forces has been formed in its special manner. In this connection we come across the important laws of "Minimum Work-variation" and "Differential coefficients of Work-variation with respect to the externally applied forces," which bear the name of the Italian engineer Castigliano.

At this point I might illustrate the "Minimum Principle" of Castigliano by the simplest possible example that lies to hand. A table with three legs is statically determined, that is to say, we can with the aid of the rules of ordinary Statics of solid bodies, determine how the applied load on the table together with its own weight is distributed among the three legs. The table with four legs is however statically undetermined. We can in fact suppose any force x we please to act in one of the four legs, and can then for the remaining three legs calculate the necessary forces, which together with x will give equilibrium with the load on the table. (Of course we suppose that the resultant of the loads does not lie in the plane of two of the legs, otherwise the proposition would be trivial). An infinite number of equilibrium distributions would thus be possible. How then does Nature make choice amongst this infinity of force systems? To this question Castigliano's Minimum Principle gives the answer. Nature favours that choice whereby she escapes with the smallest output of change in work. It may be assumed

* Described in the supplement to Dyck's Catalogue of mathematical models, apparatus and instruments, München 1893.

that the table yields slightly in a uniform manner and thus we obtain the combined alteration of work in the legs of the table, which would arise from the given loads and the reactions transmitted from the floor. Our force x is then the only unknown, for we have already expressed the stresses in the other legs in terms of x and the given loads. Thus the Work-variation will be a perfectly determined function of x of the second degree. The application of the condition for a minimum leads to a linear equation for this unknown.

I can scarcely conceive a more comprehensive and instructive method for the solution of mechanical problems. The generality is not affected if we replace our special case by any frame we please, as for example a bridge girder with redundant bars or redundant supports: or if in place of a braced system any structure subjected to elastic strains. Instead of a single linear equation we have several to determine the unknown quantities. I might here mention that the method is applicable to the determination of the displacements of the joints of a braced structure, the deflection of a beam, the torsion of a shaft, and in short to every sort of elastic strain depending on the Work Function.

As I have already had occasion to remark, Applied Dynamics was at first overshadowed by the luxuriant growth of Statical problems. To Radinger is due the great service of having awakened the dynamical conscience of the student of Applied Mechanics. He discovered afresh for machine design the Newtonian axiom, that the product of mass and acceleration is force. In reality an appreciation of this law alone, suffices to make one understand, how the mass of the forward and backward moving parts of a reciprocating engine diminishes the effect of the steam in the one phase of the throw, and in the other phase assists it; and that, in the so-called inertia effect of the pieces of a machine, we possess a means of influencing the transmission of force in an advantageous manner. In laying stress on this new knowledge Radinger has not altogether kept himself from exaggeration and he has perhaps in consequence overrated the future of high speed engines, for which the inertia effect comes most prominently into question.

I might take this opportunity of remarking that the dynamics of a Crank, which is treated synthetically in books on Applied Mechanics, in as much as the action of the forces in the separate members is investigated consecutively, at the same time furnishes the most beautiful illustration of the general analytical methods. This example has as great a degree of generality and simplicity as can be desired. We have in this instance to deal with a system with one degree of freedom, or a constrained motion for which the impressed force depends on the position of the system (*i.e.* on the position of the piston for the time being). [The effect of the regulator, which increases the number of degrees of freedom to two, is here naturally left out of consideration.] Applying to this system the analytical expression of d'Alembert's Principle, Lagrange's General Equations, or one of the other general principles of Mechanics, we obtain quite spontaneously in the different members of the resulting equations, the precise expression for the shear in the crank pin and for the inertia effect. In addition we discover, what in the elementary treatment is obscured, that the to and fro moving parts contribute to the effect of the flywheel as if half the masses of these parts were concentrated in the crank pin.

Perhaps the more constructive treatment of the crank mechanism, which is so important in applied work, is more instructive than the general analytical method. One should guard against presenting such analytical generalities to the student as an introduction to Applied Mechanics. Indeed its extraordinary power will be brought home to the mind of the *advanced* student by taking a suitable example. The teacher of Mechanics in our universities, where the analytical methods are fully investigated as an end in themselves, should not fail to take the motion of a crank as an example.

One of the most beautiful investigations in Applied Mechanics, which has been brought to light within the last decade, is connected with the inertia effect of engines: I refer to the theory of Balancing of compound engines by Otto Schlick. In this connection we are not dealing with a mechanical speciality so much as a question of a general nature, on which every one who ponders on the subject of Mechanics should form his own conclusions. In truth, it was the stern necessity of Applied Mechanics (as is so frequently the case) that timed the solution of the problem. As in the case of locomotives, so also in ship building, the question of Balancing has made its appearance. It may be stated as follows: we are given a system of masses which move in a prescribed manner. These transfer to the frames on which they work (viz. the bed of the locomotive or the hull of the ship) the forces arising from the inertia of their motion, and in consequence the frame is partly set in motion as a whole and partly strained elastically. For locomotives only the movement as a whole comes under observation: the component effects, which result when we consider the force and couple separately, are called in railway working "Zucken und Schlingern" (hammering and rolling). In ships on the other hand it is the elastic deformations which are particularly important and dangerous, if the period of the natural vibration of the ship comes near the period of the engine stroke; if, in short, as one may say, the construction of the ship is tuned to the measure of the ship's engine. In order to divert his attention from all accidental circumstances Schlick compared the vibration of the ship to the swinging of a freely moving beam on which periodic bending forces act, and studied in his beautiful models the resonance effect between the proper swing of the beam and the induced periodic force.

The frequency of a beam or of a ship diminishes with increase in length; so it appears that with continued increase in length (in which direction modern developments tend) those limits must of necessity be reached, for which the dangerous resonance effect between the period of engine stroke and natural vibration of the ship comes into play.

It appears then that ship-building must either forsake the chosen course or the inertia effects must be rendered harmless. An old political principle suggested itself—to separate the enemies into two forces and pit the one against the other. The actual means employed were provided by a special arrangement of the angle of intersection of the different pistons and a special choice of the masses and space ratios of the individual working parts. In such wise the inertia forces of the ship's engine were mutually destroyed and the hull of the ship was freed from its tormentors.

Mathematically the expression for perfect balancing (to all intents and purposes it is perfect when taken to the second order) receives the form of a law expressed by eight simple equations, in the further treatment of which Schlick was assisted by Schubert and H. Lorenz. It may be mentioned that the modern steam ships Kaiser Wilhelm der Grosse and Deutschland are provided with Schlick's balancing engines, the former with the perfect Balancing of the first order and the latter with an approximate Balancing of the second order. These masterpieces of German engineering art, which form the objects of our justifiable national pride, have become through the perfection of the theory of Balancing the best of their kind.

I now approach the last problem to which we shall allude to-day—the theory of the friction of a shaft in its bearing. As to the effects of friction there are two diametrically opposed theories, the one the theory of solid friction, which originated with Coulomb and has been already alluded to in connection with earth pressure, and the other the theory of fluid friction, which in its simplest form was stated by Newton. In Applied Mechanics the former theory is so much in vogue that it has been applied to the theory of the friction of bearings, although the presence of a fluid lubricant is absolutely essential. In accordance with the theory of the friction of solids, it is usual to assume that the magnitude of the friction couple

is proportional to the pressure on the axle, or, more precisely, is equal to the product of a coefficient of friction, the radius of the axle and the pressure on the bearing. The coefficient of friction might then be regarded as an empirical constant to be determined by experimenting with the bearing under a given load and a given period of revolution.

The theory of fluid friction was first applied to bearings by the Russian engineer Petroff. According to this theory, the whole loss of energy by friction occurs in the body of the lubricant, and is expended in rubbing the individual layers of the lubricant against one another, the bounding layers adhering to the revolving axle and the fixed bearing respectively. If we assume that the axle and bearing are accurately concentric, the friction couple is on this view proportional to the velocity of rotation and independent of the pressure. The coefficient of viscosity of the lubricant and the diameter of the bearing enter into the factor of proportionality. Further it was Osborne Reynolds who expounded more fully and improved the hydrodynamical theory of the friction of bearings, in as much as he abandoned the hypothesis of a central position for the axle: this new point of view was necessitated by the fact that the hydrodynamical friction and pressure should make equilibrium with the pressure on the axle communicated from without.

What then has experiment to say to the one or the other theory? Generally speaking the reply is as follows. For low velocities or heavy loads the magnitude of the pressure on the axle is the important factor. For high velocities or relatively small loads the friction couple is independent of the pressure on the axle. In the former case the theory of dry or solid friction represents the phenomena, while in the latter the action and behaviour are what might be expected according to the theory of fluid friction with a centrally situated axle.

I might refer to one point in connection with this. According to the solid friction theory, the axle will touch the bearing at a point, which is displaced from the line of action of the resultant pressure on the axle, in a sense opposite to that of the rotation. On the other hand, according to the hydrodynamical theory, the point of closest approach between axle and bearing and the point of greatest hydrodynamical pressure is displaced in the same sense as that of the rotation. At my suggestion this was put to the test in the early part of this year by Herr Becker, chief engineer in the railway works at Witten. Certain locomotives sent in for repair were examined for the wear of their bearings. Out of 20 bearings brought under inspection, it appeared that 16 had worn more in front, *i.e.* corresponding to the sense of rotation and in only 2 behind, while in the case of the remaining 2, the point of greatest wear was uncertain. It would seem then that the hydrodynamical theory is supported in a striking manner by this little statistical report.

These developments of Applied Mechanics, of which I have spoken, can serve no other purpose than to show that a lively scientific interest pervades the subject, that it teems with problems, is rich in difficult and elusive questions, and is laden with beautiful almost ripened fruits, which only await the skilful hand that knows how to pluck them.

The time is gone once and for all when the physicist and mathematician could superciliously hold himself aloof from the pursuits of the technical worker, because he perceived a smaller degree of scientific rigour in these branches than in his own particular sphere. Applied sciences, at least here in Germany, have developed from their own innate power a confident and self-sufficient position. We the theoretical enquirers record it to our honour, if we can assist in the building up of applied science, and we appreciate our good fortune whenever it brings us into active contact with the problems of technology.

MATHEMATICAL NOTES.

143. [D. 2. a.] *A set of criteria for convergency or divergency of series of positive terms.*

It is common to regard I. and the first part of III. of the following more complete list as giving the fullest information on the subject which is afforded without the application of special methods, based on condensation, separation or rearrangement of terms, to particular series. There is, however, a natural reluctance to accept as final anything but a full reference to all possibilities as to the value of $a_n\frac{u_n}{u_{n+1}}-a_{n+1}$. There is besides a discontent with the inequality of allusion to convergency and divergency: as to convergency, I. gives an absolute test, free from any use of knowledge previously acquired, whereas III. gives only a test as to divergency which presupposes the possession of information about the divergency of $\Sigma\frac{1}{a_n}$. The most commonly used scale of included tests is obtained by taking for a_n in turn 1, n, $n\log n$, $n\log n\log(\log n)$, ... ; and III. enables us to say that certain series examined are divergent because $\Sigma\frac{1}{n}$, $\Sigma\frac{1}{n\log n}$, ...: are divergent. But V. does not use these facts as already known. It is readily applied to prove them.

In the following, α, A, β, B mean assignable (and therefore finite) positive numerical constants, independent of n. $\Sigma u_n \equiv u_1+u_2+u_3+\ldots$ is a series of positive terms, and a_1, a_2, a_3, ... a sequence of positive quantities (a_n dependent on n) assigned in any way at convenience.

I. *If from some value of n onwards $a_n\frac{u_n}{u_{n+1}}-a_{n+1}>\alpha>0$, then Σu_n is convergent.*

II. *If from some value of n onwards $a_n\frac{u_n}{u_{n+1}}-a_{n+1}>0$, whether tending to limit 0 or not, then Σu_n is convergent in case $\Sigma\frac{1}{a_n}$ is. In case, however, $\Sigma\frac{1}{a_n}$ is divergent, Σu_n is divergent provided $a_nu_n>\beta>0$.*

III. *If from some value of n onwards $a_n\frac{u_n}{u_{n+1}}-a_{n+1}<0$, then Σu_n is divergent in case $\Sigma\frac{1}{a_n}$ is. In case, however, $\Sigma\frac{1}{a_n}$ is convergent, Σu_n is convergent provided $a_nu_n<B<\infty$.*

IV. *If from some value of n onwards*

$$a_n\frac{u_n}{u_{n+1}}-a_{n+1}<-\alpha<0, \text{ and } a_nu_n<B<\infty,$$

then Σu_n is convergent.

V. *If from some value of n onwards*

$$a_n\frac{u_n}{u_{n+1}}-a_{n+1}<0, \text{ but } >-A>-\infty,$$

and if a_nu_n tends to infinity with n, then Σu_n is divergent.

Cases in which these criteria give no information are:

(1) when $a_n\frac{u_n}{u_{n+1}}-a_{n+1}$ has not always the same sign for large values of n;

(2) when $a_n \frac{u_n}{u_{n+1}} - a_{n+1} > 0$ but has limit 0, and there is doubt as to the convergency or divergency of $\Sigma \frac{1}{a_n}$;

(3) when $a_n \frac{u_n}{u_{n+1}} - a_{n+1} > 0$ but has limit 0, and $a_n u_n$ has limit 0, and $\Sigma \frac{1}{a_n}$ is divergent;

(4) when $a_n \frac{u_n}{u_{n+1}} - a_{n+1} < 0$ but has limit 0, and $a_n u_n < B < \infty$, and there is doubt as to the convergency or divergency of $\Sigma \frac{1}{a_n}$;

(5) when $a_n \frac{u_n}{u_{n+1}} - a_{n+1}$ tends to $-\infty$, and $a_n u_n$ tends to infinity, and $\Sigma \frac{1}{a_n}$ is convergent or doubtful.

The overlapping of certain criteria gives the following information:

(i) If $a_n \frac{u_n}{u_{n+1}} - a_{n+1} > \alpha > 0$, and $a_n u_n > \beta > 0$, then $\Sigma \frac{1}{a_n}$ is convergent;

(ii) If $a_n \frac{u_n}{u_{n+1}} - a_{n+1} < -\alpha < 0$, and $a_n u_n < B < \infty$, then $\Sigma \frac{1}{a_n}$ is convergent.

Of the five main criteria, II. and III. are only tests by comparison. They express in other terms, taking $\frac{1}{a_n}$ for c_n or d_n, that, if Σc_n is a convergent and Σd_n a divergent series of positive terms, Σu_n is convergent if from some n onwards the ratio $\frac{u_n}{c_n}$ never surpasses an assignable numerical magnitude, and on the other hand is divergent if $\frac{u_n}{d_n}$ never falls below an assignable magnitude greater than zero.

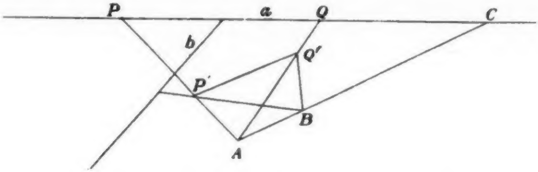
The establishment of the other tests, which have an absolute character, depends secondarily on these laws of comparison but primarily on the one fundamental fact as to convergency and divergency that $\Sigma(v_n - v_{n+1})$ is convergent or divergent according as the sequence $v_1, v_2, v_3, \ldots v_n, \ldots$ does or does not tend to a limit as n increases. If, for instance (to take IV. and V.), $v_n - v_{n+1}$ is always negative from $n = r$ onwards, so is $v_r - v_{n+1}$ by addition. No term of the sequence $v_{r+1}, v_{r+2}, \ldots$ is so small as v_r. Also the terms of the sequence increase as we go on in it, since every $v_n - v_{n+1}$ is negative. If all are less than some B, there must be a definite quantity not greater than B which separates those quantities between v_r and B which v_n can be made to exceed, by increasing n sufficiently, from those which it cannot; and the sequence $v_1, v_2, v_3, \ldots$ tends to this limit. If no such B can be assigned it tends to no limit. Thus, taking $a_n u_n$ for v_n, there is convergency of $\Sigma(a_n u_n - a_{n+1} u_{n+1})$ under the circumstances of IV. but not under those of V. Applying then the laws of comparison to the negative of this series, regarded as Σc_n or Σd_n, and the series Σu_{n+1}, the assertions of IV. and V. follow.

E. B. ELLIOTT.

144. [P. 3. b. a.] *To find the relation between two maps of the same contour on the stereographic projection.*

Let P' be any point of the curve σ lying on a sphere whose centre is O. Let α, β be the projections of σ from any two points A, B of the sphere on to the planes a, b through O parallel to the tangent planes at A and B. Let C be the projection of B from A on a. Rotate b about the intersection l of a and b through an angle AOB till it coincides with a. Then β is derived

from a by inverting with respect to a circle of radius $CA(=OA \operatorname{cosec} \frac{1}{2}AOB)$ whose centre is C, and then reflecting in l.



For let Q' be the reflexion of P' in the plane bisecting AB at right angles, and let P, Q be the projections of P', Q' from A on a. Then

$$2OA^2 = AP \,.\, AP' = AQ \,.\, AQ',$$

and therefore the angle $QPA = P'Q'A = QAC$ (since $P'Q'$ is parallel to AB). Therefore $CP \,.\, CQ = CA^2$; and by symmetry Q is the reflexion in l of the projection of P' from B on b when the plane b has been brought to coincide with a by rotation about l.

HAROLD HILTON.
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144. [L^1. 1.] *Proofs of theorems in geometrical conic sections.*

S, S' are the foci, C the centre, A the vertex corresponding to S, and X the point where SA meets the directrix corresponding to S.

1. Given S, A, and X, find the points at which a straight line l parallel to SX meets the conic.

Draw XM, AR perpendicular to SX to meet l in M and R. Let SM meet AR in Q. Draw a circle with centre Q and radius QR. The points in which the tangents from S to this circle meet l are the points required.

The fact that a conic is symmetrical about two straight lines and many such elementary theorems follow at once from this construction.

2. Given S, A, and X, find the points at which any straight line QF meets the conic.

Draw QK parallel to SX, and XFK perpendicular to SX. Take a length D such that $D : QK = SA : AX$. Divide FS internally and externally at R and R' in the ratio $FQ : D$. The circle on RR' as diameter cuts QF in the required points.

3. From a point T tangents TY, TZ are drawn to a conic, show that the angles STY, $S'TZ$ are equal.

Draw SY, SZ perpendicular to the tangents. Bisect ST at R. Since R and C are each equidistant from Y and Z, CR is perpendicular to YZ, and therefore $S'T$ is perpendicular to YZ. Hence

$$S'TZ = 90° - TZY = 90° - TSY = STY.$$

4. If the tangents are at right angles, find the locus of T. Construct as in (3). Then

$$2CT^2 = ST^2 + S'T^2 - 2CS^2 = 4YR^2 + 4CR^2 - 2CS^2$$
$$= 2CY^2 + 2CZ^2 - 2CS^2 = 4CA^2 - 2CS^2.$$

Therefore T lies on a circle whose centre is C.

This method also gives the locus of the intersection of two perpendicular lines, each of which touches one of two fixed confocal conics.

5. If CP, CD are conjugate semi-diameters of an ellipse, $SP \,.\, S'P = CD^2$.

$$2SP \,.\, S'P = (SP + S'P)^2 - SP^2 - S'P^2 = 4CA^2 - 2CP^2 - 2CS^2 = 2CD^2.$$

This proof may be modified to suit the case of a hyperbola.

Univ. Coll., Bangor. HAROLD HILTON.

145. [L². 10. g.] *A circle and sphere connected with a confocal system of conics and a confocal system of conicoids respectively.*

I. The well-known equation for the squares of the semi-minor axes of the two conics of a confocal system which pass through any point $x'y'$ is

$$\lambda^2 - \lambda(x'^2 + y'^2 - a^2e^2) - a^2e^2y'^2 = 0. \quad \text{(See Smith's \textit{Conics}.)}$$

A geometrical theorem follows at once.

For the sum of the roots is $x'^2 + y'^2 - a^2e^2$.

Hence the sum will be positive, zero, or negative, according as the point $x'y'$ is outside, on, or inside the circle $x'^2 + y'^2 - a^2e^2 = 0$.

But b_2^2, the square of the minor axis of the hyperbola, is the negative analytical quantity, and is the negative of the square on the geometrical minor axis. Hence the circle on SS' as diameter divides those points, which are such that of the two conics through them the minor axis of the ellipse is greater than the minor axis of the hyperbola, from those points having the converse property, while for points on the circle itself the minor axes of the ellipse and hyperbola are equal.

II. Again for a system of conicoids the equation that gives the squares of the least axes of the three conicoids through any point x', y', z' is

$$\frac{x'^2}{a^2 - c^2 + \lambda'} + \frac{y'^2}{b^2 - c^2 + \lambda'} + \frac{z'^2}{\lambda'} = 1.$$

Hence the sum of the roots is

$$x'^2 + y'^2 + z'^2 - (a^2 + b^2 - 2c^2),$$

and in a similar way the points, which are such that of the three conicoids of the system through them the square on the least axis of the ellipsoid is greater than the sum of the squares on the geometrical least axes (viz. the negative of the analytical squares) of the two hyperboloids, are separated from those points for which it is less by the sphere concentric with the system and passing through the angular points of the square formed by drawing parallels through each of the foci in* each of two pairs of foci of the system that lie in the plane containing the two larger axes of all the ellipsoids of the system, lines parallel to the join of the other pair; and on the sphere itself there is equality.

H. L. TRACHTENBERG.

Trinity Coll., Cambridge.

146. [R. 5. a.] *To prove that the circles on one side of the radical axis of any given non-intersecting coaxial system can be described simultaneously by a swarm of particles under the attraction of a central force.*

The above theorem was devised after the converse of Hamilton's theorem on the law of force in a conical orbit had been demonstrated to me. This converse is: The possible orbits under a central force varying directly as the distance from the centre of force and inversely as the cube of the distance from a fixed straight line are conics having the centre of force and the line for pole and polar. There is one point to be noticed, viz. orbits which are integrals of the equations of motion that are cut by the line in real points must be rejected, as by reference to the law of force it is apparent that at such points the force becomes infinite and the equations of motion used are equations of motion under assumption of finite force and preclude such a case. To prove the above theorem we see that if we take the limiting point on the same side of the radical axis as centre of force, and the line parallel to the radical axis through the other limiting point for the given line, then since by a theorem in geometry all these circles have this point and line for pole and polar, and since the line does not cut any in real points, they are possible orbits.

There remained only one point to settle.

* It will easily be seen which two of the three pairs of foci are meant.

From the equation of the orbits given me it was apparent that all conics having the centre of force and given line for pole and polar were not obtained unless we took

(1) all orbits described under an attractive force of above magnitude ;

(2) all orbits described under a repulsive force of above magnitude.

But I deduced from that equation that the circles under consideration came under case (1).

Hence the theorem that the circles can be simultaneously described under an attractive central force is true. H. L. TRACHTENBERG.

Trinity Coll., Cambridge.

147. [**V. 1. a.**]. *On decimalisation of money.* (Cf. Note 140, p. 383, Vol. II.)

Shillings and sixpences can be expressed accurately as decimals of a pound.

$\frac{1}{4}$d. = £·001 + ·01d. accurately.

To multiply £3. 9s. $7\frac{1}{2}$d. by 365.

£3. 9s. $7\frac{1}{2}$d. = £3. 9s. 6d. + 6 f. = £3·475 + £·006 + ·06d.
= £3·481 + ·06d.

£	d.
3·481 +	·06
365	
1044·3	21·90
208·86	
17·405	

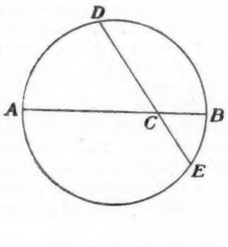
£1270·565 + 21·90d.
= (£1270. 11s. + £·015) + (1s. 9·75d. + ·15d.)
= £1270. 11s. $3\frac{3}{4}$d. + 1s. $9\frac{3}{4}$d.
= £1270. 13s. $1\frac{1}{2}$d.

This work can be abbreviated. I have found it valuable as a help to accuracy (*a*) to look for a £·015 in the first product (after sixpences have been removed), (*b*) to then look for a ·15d. in the second part.

E. E. CHAMBERS.

148. [**X. 4. b. a.**] *A graphical solution of the typical quadratic equation* $ax^2 \pm bx \pm c = 0$.

(i) The fact that the rectangles contained by the segments of intersecting chords of a circle are equal leads to a very simple graphical solution of the equations of the form $ax^2 \pm bx + c = 0$ where a, b, c are all positive quantities.



(*a*) When $b^2 - 4ac$ is positive or the roots are real.

Take any two numbers p, q whose product is $\frac{c}{a}$ and sum greater than $\frac{b}{a}$.

Draw ACB such that $AC = p$ and $CB = q$.

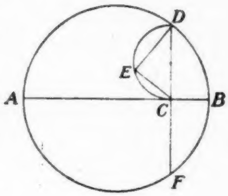
On AB as diameter describe a semicircle and through C by the ordinary geometrical method draw a chord DCE of length $\frac{b}{a}$. Then DC and CE represent the roots of $ax^2 - bx + c = 0$ and $-DC$ and $-CE$ the roots of

$ax^2 + bx + c = 0$.

For $DC \,.\, CE = AC \,.\, CB = p \,.\, q = \frac{c}{a}$ and $DC + CE = \frac{b}{a}$.

(β) When b^2-4ac is negative or the roots are imaginary.

As before, take any two numbers p, q whose product is $\frac{c}{a}$ and sum greater than $\frac{b}{a}$. Draw ACB such that $AC=p$ and $CB=q$. On AB as diameter describe a semicircle, and it will now be found that all chords drawn through C are greater than $\frac{b}{a}$. Hence we conclude the roots are imaginary and proceed as follows.



Through C draw the chord DCF perpendicular to AB, and on DC as diameter describe a circle. From D draw the chord DE equal to $\frac{b}{2a}$ and join EC.

The roots of $ax^2-bx+c=0$ will be $DE+\iota\,.\,EC$ and $DE-\iota\,.\,EC$ where 2 represents $\sqrt{-1}$.

Now $$ED=\frac{b}{2a} \text{ and } DC^2=AC\,.\,CB=p\,.\,q=\frac{c}{a}.$$

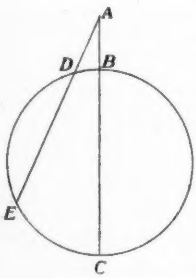
Hence $$EC^2=DC^2-ED^2=\frac{c}{a}-\frac{b^2}{4a^2}=\frac{4ac-b^2}{4a^2};$$

$$\therefore\ -EC^2=\frac{b^2-4ac}{4a^2};$$

$$\therefore\ \iota\,.\,EC=\frac{\sqrt{b^2-4ac}}{2a};$$

$$\therefore\ DE\pm\iota\,.\,EC=\frac{b\pm\sqrt{b^2-4ac}}{2b},$$

that is, $DE\pm\iota\,.\,EC$ represent the roots of $ax^2-bx+c=0$.



(ii) The corresponding proposition for chords intersecting outside the circle enables us to solve $ax^2\pm bx-c=0$.

Take any two numbers p, q whose product is $\frac{c}{a}$, provided their difference is greater than $\frac{b}{a}$.

Draw ABC so that $AC=p$ and $AB=q$.

On BC as diameter describe a circle and from A draw a secant ADE such that DE is equal to $\frac{b}{a}$.

Then AE and $-AD$ are the roots of $ax^2-bx-c=0$ and $-AE$ and AD are the roots of $ax^2+bx-c=0$.

For $-AE\,.\,AD=-AB\,.\,AC=-p\,.\,q=\frac{c}{a}$ and $AE-AD=DE=\frac{b}{a}$.

W. O. HEMMING.

REVIEWS.

A Treatise on the Line Complex. By C. M. JESSOP. Pp. xvi, 364. 10s. 1903. (Cam. Univ. Press.)

This is a welcome addition to the number of books in the English Language dealing with modern developments of pure mathematics. It is in fact the only

book on line-geometry in English, the only other general account of this subject in English being Mr. J. H. Grace's valuable article in the *Encyclopaedia Britannica*, 10th edition, vol. 28, pp. 659-664.

In ordinary geometry of two or three dimensions points are the elements, and equations represent curves or surfaces regarded as loci of points. Again, when plane-coordinates (tangential coordinates) are employed for three-dimensional space, planes are the space-elements, and an equation represents a surface regarded as the envelope of a system of planes. So line-geometry is geometry of space in which straight lines are the elements. It is evident from the point equations of a straight line in space that four independent quantities must be specified to define a straight line; these four quantities may be treated as the coordinates of a straight line. Space therefore contains ∞^4 straight lines.

After a short introduction dealing with Double Ratio, Correspondences, etc., the author commences with the different systems of coordinates of lines. It is found advisable to use 6 homogeneous coordinates, which are connected by a quadratic identity. Plücker, who originated the subject, used the six coordinates $p_{12}, p_{13}, \ldots$, defined as follows: $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $(\beta_1, \beta_2, \beta_3, \beta_4)$ being the point-coordinates of any two points on the line, then the coordinates of this line are:

$$p_{12} = \alpha_1\beta_2 - \alpha_2\beta_1 \text{ etc., and the identity is}$$

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

Klein introduced a great improvement by a transformation of these coordinates, by which he obtained the system of coordinates x_1, x_2, x_3, x_4, x_5, and x_6, where $\Sigma x^2 = 0$. The author mainly employs the analytical method with these coordinates, the symmetry of which much simplifies the analysis. The methods of synthetic geometry however are used where they are found appropriate.

One equation in these coordinates represents a triply infinite system of straight lines, which is called a Complex. For instance, the system of all the straight lines which intersect a given straight line is a special kind of the linear complex.

Two equations represent a doubly infinite system, which is called a Congruence. The lines of a Congruence are all bitangents of some surface. If both equations are linear, the congruence consists of all the straight lines intersecting two given straight lines.

Lastly, three equations represent a singly infinite system of straight lines, that is, a Ruled Surface. If the three equations are all linear, the Ruled Surface is a quadric, or more accurately, one of the two sets of generators of a quadric (which set the author calls a Regulus); from the line-geometry point of view, the other set of generators is a distinct thing, and it is only from the point-geometry point of view that the two Reguli become one surface. As is well known, a quadric is generated by the system of straight lines which meet three given straight lines. Evidently the properties of ruled surfaces as such can best be studied by means of line-geometry.

The author deals fully with the properties of the Linear Complex and of the Quadratic Complex. There are no less than 55 distinct species of the latter complex. Numerous interesting results are also obtained relating to Congruences and Ruled Surfaces, but they are hardly capable of being summarised within the limits of a review.

The author gives a good account of the analogies between line-geometry and other four-dimensional geometries. Perhaps the most remarkable of these is the connexion, discovered by Sophus Lie, between line-geometry and sphere-geometry. Lie showed that a correspondence can be established between the two geometries, in which one sphere corresponds to one straight line, and two spheres which touch correspond to two straight lines which intersect. On this subject reference may be made to Lecture II. of Klein's charming *Lectures on Mathematics* (Macmillan, 1894). The final chapter of Mr. Jessop's book deals with partial differential equations connected with the Line Complex.

The book contains, as its title implies, a fairly complete account of the subject as at present developed, and it therefore contains much that will be of interest only to the specialist. The elements of line-geometry however are not difficult, and are of quite as much interest and importance as some of the advanced parts of ordinary Solid Geometry.

A. E. WESTERN.

The Constructive Development of Group-Theory. By B. S. EASTON. Pp. iv, 89. University of Pennsylvania. 1902.

This is a bibliography and a catalogue of results, classified and tabulated. It aims at completeness, and is at any rate remarkably full and minute in its bibliographical section of 36 pages. It is the record of an examination of all the literature of Abstract and Substitution Group-Theory. The theory of linear groups is not included, nor that of continuous transformation groups. The labour expended on the production must have been great, and a very compendious work of reference has resulted. The author's prefatory note speaks of his purpose having been to construct from the material collected a continuous treatise on the subject. The result, however, is a syllabus rather than a treatise, and its atomistic character is more conspicuous than its continuity. No proofs are given, their omission being accounted for by need to economise space, and the very few literary passages—as for instance that beginning on p. 43, which speaks of the separation of abstract group-theory from that of substitution groups as not easy to mark, and proceeds to say a little more without attempting to remove the difficulty alluded to—are of slight explanatory value. The student who does not ask for exposition, but has mastered the leading ideas of the subject and wishes for guidance as to where to study more of it, will find exactly what he needs in these pages. They are richer in reference, for instance, than is the article on the subject in the *Encyclopädie der Mathematischen Wissenschaften.* The compilation seems to be the outcome of academic encouragement and of sustained effort on the part of a young researcher, and does credit both to him and to the University of Pennsylvania.

Les Mécanismes : Traité élémentaire de cinématique appliquée. H. LEBLANC. Paris : Garnier Frères, 1903. Pp. 432.

Professor Huxley in the autobiographical sketch prefixed to his essays remarks that he always considered himself as a mechanical engineer *in partibus infidelium.* Inverting this idea it may be suggested that the mental satisfaction derived from the study of mechanisms is akin to that of the zoologist, for it arises from the contemplation of the almost infinite variety of results attained by variations on a very limited number of primitive forms.

For example, we recognise in the mechanism for enabling a tricycle, car, or traction engine, to turn a corner, and in certain two-speed gears and dynamometers, a combination of two pairs of co-axial bevel wheels whose axes are at right angles, which is known as the differential bevel gear. This was first employed in cotton spinning machinery. If thread, delivered at a uniform rate, is wound on to a bobbin which revolves at a uniform rate trouble will arise, because the diameter of the bobbin increases as the thread winds on to it. It was to obviate this difficulty that the differential gear, which has since played so many parts, was invented.

M. Leblanc's book should prove of great service to anyone who takes a pleasure in following out such a comparison. He has brought together some 200 mechanisms, comprising examples of brakes, safety clutches, friction clutches, change speed gears, steering gears, and free-wheel mechanisms, each illustrated by a diagram and accompanied by a concise yet wonderfully clear description.

No student of machinery can fail to derive the satisfaction alluded to, or to obtain several interesting specimens worthy of close examination, from the collection of M. Leblanc. C. S. JACKSON.

Recueil d'Exercices sur le Calcul Infinitésimal. By M. F. FRENET. Sixth Edition. 8 frcs. (Gauthier-Villars.)

The first edition of this collection of examples and their solutions appeared in 1856 and consisted of but 220 pages. It was constructed avowedly on the lines of a book well known in those days and even now of interest from the historical point of view—*Gregory's Examples.* Being intended primarily for the use of the candidates for the École Polytechnique and École Normale it has considerably grown with the steady expansion of the syllabus of the examinations at those institutions. It now consists of over 500 pages and contains about 120 pages more than appeared in the last edition. This is largely due to the inclusion of the theory of elliptic functions and the theory of the functions of a variable

imaginary among the new subjects of the official programme. Questions dealing with residues, elliptic functions, partial differential equations, etc., have been added by M. H. Laurent, and the whole has been thoroughly revised and brought up to date.

Notions de Mathématiques. By JULES TANNERY. **Notions Historiques.** By PAUL TANNERY. Pp. x, 352. 1903. (Delagrave.)

This little book is an endeavour to meet the difficulty caused by the extension of the course of mathematics taken by the "classe de Philosophie." The additions to that course are such as have hitherto been read by students giving a considerable portion of their time to analysis and geometry. The author aims to present the subject in such a general way as to make intelligible to the student of philosophy the methods and processes of mathematics, so that he may find that in a few years' time the language of the science is not to him an unknown tongue, and so that he will be the better able to grasp an idea of scientific progress and the applications of science which are, as the author says in his preface, "tending day by day to modify more and more profoundly our modes of life and thought." The book may therefore be considered as an introduction to an "étude réfléchie" of the subject. It gives the student an idea of the field covered by the science, the nature of the problems which it attacks and solves, the nature of a function, its variations and its graph, the mutual relationship of algebra and geometry, the relations of number and space, and so forth. One third of the book is devoted to the elements of the subject. We cannot help thinking that most teachers will learn something from a volume written with the aim that is carefully laid down in the "conseils généraux" given in the programme for the "classe de Philosophie." "The teacher must not forget that the pupils are not accustomed to the study of the subject. He must therefore avoid all abstract theory; he will not bring general ideas before his classes, but must utilise particular examples in order to throw general ideas into relief. The syllabus is to a large extent elastic, and the master may extend its scope here and there according to the aptitude of his pupils and the interest he has succeeded in exciting in them. This refers in particular to such applied mathematics as occurs in the syllabus, and this branch of the subject should be treated broadly and without too much rigour. Historical sketches should form part of the teaching. For instance, the pupils should be told how Euclid, Archimedes, etc., made use of the method of exhaustion, and they should have some notion of how and when the differential and integral calculus came into being. The object of the teacher should be to contribute to the philosophical development of the pupils, by instilling into them important ideas; and on the other hand he must furnish them with the concepts which they will need should they ever require to take their certificates in physics, chemistry, or any other form of natural science." With these objects in view it cannot be that the book is without interest to the British teacher of Mathematics. We only add that the 25 pages given by M. P. Tannery to his historical sketch are all too short.

A First Course in Infinitesimal Calculus. By D. A. MURRAY. Pp. xvii, 439. 1903. (Longmans, Green.)

Professor Murray has written an introduction to the infinitesimal calculus for those whom circumstances compel to study without the guidance and assistance that a teacher can afford. We have no hesitation in saying that it is admirably adapted for this as well as for ordinary school purposes. The author is evidently a teacher of great experience, and is familiar with the special pitfalls which beset the path of the ordinary student in approaching this subject. The best teacher is he who can anticipate the difficulties and smooth the path of the student without however removing every obstacle. The sooner the student learns that mathematics is a hard subject the better. He must be prepared to find that there is much to be grappled with, and it is by boldly facing the difficulties and trying to understand them without undue assistance that the intellectual fibre is strengthened, and the study of mathematics becomes a mental training of incomparable value. In his preface, Professor Murray takes the unusual course of expressing his obligations to "many students whose difficulties and original opinions have interested and stimulated me." But the author has not made the mistake of making things too easy for his classes. And for the sake of those

whose interest in the development of the various sections has been excited he has supplied copious references to the best text-books in England and America. While there is nothing novel in the content of the book, the author is to be congratulated on the skill with which the material is arranged, and the careful manner in which the pupil is led on from stage to stage of the subject.

Mathematical and Physical Papers. By the late Sir G. G. STOKES, Bart. Vol. IV. Pp. viii, 378. 1904. (Cam. Univ. Press.)

The fourth volume of these papers includes those published between 1853 and 1876. They have been carefully selected by Professor Larmor with the assistance of Lords Kelvin and Rayleigh and Mr. F. G. Hopkins, "only those memoirs and notes involving distinct additions to scientific knowledge being included." Among the mathematical papers are "On the Discontinuity of Arbitrary Constants which appear in Divergent Developments," with a supplementary paper; "On the Internal Distribution of Matter which shall produce a given Potential at the surface of a Gravitating Mass"; "On the Communication of Vibration from a Vibrating Body to a surrounding Gas." The beautiful portrait prefixed to the volume is taken from Mr. Dickinson's painting in Pembroke College. The next volume, which is in the hands of Lord Rayleigh, will contain the remaining papers, together with a biographical notice, which will be looked forward to with great interest. A glance at the contents of this volume is sufficient to show the many-sided versatility of this great *savant*. The account of his life and work could not be placed in better hands than in those of Lord Rayleigh.

Lehrbuch der Analytischen Geometrie. Bearbeitet von O. FORT und O. SCHLÖMILCH. Erster Teil: **Analytische Geometrie der Ebene** Von. O. FORT. Siebente Auflage. Besorgt von R. HEGER. Pp. xvii, 268. 1904. (Teubner.)

This volume has changed but little since its first appearance nearly half a century ago. It is primarily intended for the students at the Königl. Sächs. Polytechnikum at Dresden, and rather more attention than usual is paid to questions that are capable of graphical interpretation or application. For instance, we find an account of the generation of the conics by the intersection of projective pencils. Conics are treated at first individually and in detail so as to bring out particular geometrical properties; then, starting with the equation of the second degree, they are discussed and reduced to their simplest forms and incidentally, as it were, we are introduced to the general properties of the curves of the second and higher orders. The proofs are well arranged and simply put. There is no direct reference to the ideas of infinitesimal analysis. The book is beautifully printed. It concludes with the treatment of Cissoid, Lemniscate, Spirals, etc.

Précis d'Algèbre et de Trigonometrie à l'usage des élèves de Mathématiques Spéciales. By G. PAPELIER. Pp. 354. 1903. (Nony.)

This "precis" is a masterly arrangement of exactly what is necessary for the examinations of the students for whom it is intended. Although it gives the general impression of being the irreducible minimum, it is surprising to find what a wide extent of ground is covered. M. Papelier writes in a clear and attractive style.

First Lessons in Observational Geometry. By Mrs. W. N. SHAW. Pp. x, 148. 1903. (Longmans, Green.)

This admirable little book is the outcome of the conviction held by its accomplished authoress that the want of aptitude which we often think is shown by pupils for the study of geometry is more apparent than real. Failure is due to the uninviting manner in which the subject has been presented at the beginning. The essential object in the teaching of geometry is that the student shall eventually gain clear perceptions of the relationships of space. The students who can write out any amount of bookwork with the ease that is born of a good memory, quick handwriting, and constant practice, may nevertheless know very little of geometry at the conclusion of their studies, and it may be taken for granted that they will never of their own accord be induced to take up the subject again. "I found that even students who had reached a relatively high standard of success in mathematical examinations were still very uncertain about geometrical facts." Could there be any more biting criticism of the system of teaching which those

examinations represent? Although we do not greatly care for the way in which Mrs. Shaw has chosen to tell her tale—in a series of conversations with Jane and William, etc.—we have nothing but admiration for the exceedingly clear and simple manner in which she develops the subject.

Grundlagen der Geometrie. By D. Hilbert. 2nd Edition. Pp. 175. 1903. (Teubner.)

The second edition of Prof. Hilbert's epoch-making work on the foundations of Geometry has doubled in size owing to the inclusion of several papers from the Math. Ann. and the Proceedings of the London Mathematical, and Transactions of the American Mathematical Society.

Introduction to Quaternions. By the late Professors P. Kelland and P. G. Tait. 3rd Edition. Prepared by C. G. Knott. Pp. 208. 1904. (Macmillan.)

Dr. Knott has greatly improved on the second edition of Kelland and Tait's well-known introduction to Quaternions. He has swept away the hints and solutions which were placed at the end of that edition, and he has either deleted altogether or placed among the examples the simpler geometrical illustrations which are nowadays but child's play to the well prepared candidate for University Scholarships. Chapters III. and IV. are new. The quaternion is defined as the complex number which measures the ratio of two vectors, and the associative law is assumed to hold good in product combinations. From these two principles Dr. Knott claims that the whole of Hamilton's vector algebra evolves, and moreover that O'Brien's difficulty as to the identification of vector and quadrantal versor is removed by this method of presentation. The extension of vector ideas and notations into text-books on mechanics and mathematical physics in general has led to the inclusion of a section on dynamical applications in Chap. IX. The last four articles of the same chapter introduce the student to the mysterious Nabla. Dr. Knott was a pupil of both the authors, and a colleague and friend of Professor Tait. We need only add that the work of preparing a new edition could hardly have been entrusted to more competent hands. The book is certainly the most attractive introduction to a fascinating subject. We could have wished that Dr. Knott had seen his way to give the reader some account of the history of the subject, including some information about the genius to whom we are indebted for its invention, and the steps by which he was led to the same. With the exception of this complaint we have only to congratulate Dr. Knott on the way in which his labour of love has been performed. Looking at the subject from the point of view of the teacher, we have always been struck by the plea advanced by Kelland in his preface to the first edition. The subject of Quaternions "belongs to first principles and is their crowning and completion. It brings those principles face to face with operations, and thus not only satisfies the student of the mutual dependence of the two, but tends to carry him back to a clear apprehension of what he had probably failed to appreciate in the subordinate sciences. Besides, there is no branch of mathematics in which results of such wide variety are deduced by one uniform process . . . and what is of the utmost importance in an educational point of view, the reader . . . does not require to encumber his memory with a host of conclusions already arrived at in order to advance. Every problem is more or less self-contained." And again, it is worth while quoting Tait's remarks in the preface to the second edition of his elementary treatise. "It must always be remembered that Cartesian methods are mere particular cases of Quaternions, where most of the distinctive features have disappeared, and that, when in the treatment of any particular questions scalars have to be adopted, the Quaternion solution becomes identical with the Cartesian one. . . . It appears to me that the study of a mathematical subject like Quaternions, which, while of immense power and comprehensiveness, is of extraordinary simplicity, and yet requires constant thought in application, would also be of great benefit. With it there can be no 'shut your eyes and write down your equations,' for mechanical dexterity of analysis is certain to lead at once to error. . . ."

Five-Figure Tables of Mathematical Functions. By J. B. Dale. Pp. vi, 92. 1903. (Arnold.)

Mr. Dale's set of tables will be found of service to students of applied mathematics

and physics. In addition to the ordinary tables of logarithms, natural trigonometrical functions and the like, we find the following:—A table of zonal surface harmonics giving the first seven harmonics for values of the argument x at intervals of 0·01; tables of Bessel Functions giving $J_n(x)$ and $I_n(x)=\iota^{-n}J_n(\iota x)$; the values of $\log_{10}\Gamma(x)=\log_{10}\int_0^\infty e^{-t}t^{x-1}dx$ for values of x between 1 and 2, and $\log\Gamma(x+1)$ from $x=1$ to $x=100$; logs of factorials; and binomial coefficients for interpolation by differences. The collection will hold its own among the others in the market.

Methodisches Lehrbuch der Elementar-Mathematik. By G. Holzmuller. Vol. III. 2nd Edition. Pp. xiv, 370. 1903. (Teubner.)

The sections into which this book is divided are concerned respectively with Higher Geometry, Stereometry, Spherical Trigonometry, Elementary Algebraic Analysis, and equations of the third, fourth, and nth degrees. An important feature is the number of applications to questions in physics, astronomy, geodesy, navigation, steam, etc., and for this reason it is worth the attention of the enterprising teacher.

Der Geometrische Vorkursus in Schulgemaszer Darstellung. By E. Wienecke. Pp. 97. 1904. (Teubner.)

This is an excellent introduction to experimental geometry. There is a full course of *vivâ voce* questions which should help to sustain interest and to enable the student to form sound ideas.

Formulaire de Mathématiques Spéciales. By G. Papelier. Pp. 217. N.p. 1904. (Vuibert & Nony.)

This is a handy little volume in cloth, interleaved with blank sheets. The ground covered includes Algebra, Trigonometry and Analytical Geometry of two and three dimensions, with the applications of the calculus. Perhaps the simplest way of shewing the reader what he may expect to find in the book is to give a random specimen:

Equations générales de Coniques (p. 103).

Equations ponctuelles générales de coniques:

Coniques passant par les points A, B, C, D. (AB, $P=0$; CD, $Q=0$).

$$f+\lambda PQ=0.$$

Coniques tangentes en A à la conique $f=0$ et passant par les points B, C.

$$f+\lambda PQ=0.$$

Coniques osculatrices en A à la conique $f=0$ et passant par le point B.

$$f+\lambda PQ=0.$$

Coniques surosculatrices en A à la conique $f=0$. $f+\lambda P^2=0$.

Coniques bitangentes en A et B à la conique $f=0$. $f+\lambda P^2=0$.

Coniques homothétiques à la conique $f=0$ et passant par les points A et B.

$$f+\lambda P=0.$$

Coniques homothétiques et concentriques à la conique $f=0$. $f+\lambda=0$.

Coniques circonscrites au quadrilatère $ABCD$ (AB, $P=0$; BC, $R=0$; CD, $Q=0$; DA, $S=0$). $PQ+\lambda RS=0$.

Coniques circonscrites au triangle ABC et tangentes en A à la droite $AT(S=0)$.

$$PS+\lambda QR=0.$$

Coniques tangentes en B et C aux droites AB, AC. $QR+\lambda P^2=0$.

Coniques circonscrites au triangle ABC. $\Sigma\lambda QR=0$ ou $\Sigma\frac{\lambda}{P}=0$.

Coniques inscrites au triangle ABC. $\sqrt{\lambda P}\pm\sqrt{\mu Q}\pm\sqrt{\nu R}=0$.

Coniques conjuguées par rapport au triangle ABC. $\Sigma\lambda P^2=0$.

Coniques bitangentes aux deux coniques $f=0$, $f_1=0$, $\mu^2P^2+2\mu(\lambda_1 f_1-f)+Q^2=0$.

λ_1 désigne une racine de l'équation en λ, et P, Q deux fonctions linéaires correspondantes, telles que $f+\lambda_1 f_1\equiv PQ$. On a trois séries de coniques relatives aux racines de l'équation en λ.

Coniques inscrites dans une quadrilatère, $\frac{P^2}{\lambda}+\frac{Q^2}{1-\lambda}-R^2=0$; les équations des côtés étant $P\pm Q\pm R=0$, et celles des diagonales $P=0$, $Q=0$, $R=0$.

SOLUTIONS.

422. [K. 1. a.] *Three collinear points are at distances a_1, a_2, a_3 from one another and r_1, r_2, r_3 from a fourth point distant p from their line; prove that*

$$\Sigma a = 0, \quad \Sigma ar^2 + \Pi a = 0, \quad p^2 \Pi a^2 = 4s\Pi(s - ar),$$

where $$2s = \Sigma ar.$$

R. W. H. T. Hudson.

Solution by W. F. Beard.

Let x_1; x_2, x_3 be distances of the three collinear points from the foot of the perpendicular from the fourth point on the line, so that

$$a_1 = x_2 - x_3, \quad a_2 = x_3 - x_1, \quad a_3 = x_1 - x_2.$$

(1) $\Sigma a = 0$, obviously.

(2) $r_1^2 = p^2 + x_1^2$, etc.;

$$\therefore\ \Sigma ar^2 = p^2 \Sigma a + \Sigma x_1^2 (x_2 - x_3)$$
$$= -\Pi(x_2 - x_3)\Pi = -a;$$
$$\therefore\ \Sigma ar^2 + \Pi a = 0.$$

(3) $16s(s - a_1r_1)(s - a_2r_2)(s - a_3r_3)$

$$= -\Sigma a^4 r^4 + 2\Sigma a_2^2 r_2^2 a_3^2 r_3^2$$
$$= -\Sigma a_1^4 (p^2 + x_1^2)^2 + 2\Sigma a_2^2 a_3^2 (p^2 + x_2^2)(p^2 + x_3^2)$$
$$= -p^4 \{\Sigma a_1^4 - 2\Sigma a_2^2 a_3^2\} - 2p^2 \{\Sigma a_1^4 x_1^2 - \Sigma a_2^2 a_3^2 (x_2^2 + x_3^2)\}$$
$$- \Sigma a_1^4 x_1^4 + 2\Sigma a_2^2 a_3^2 x_2^2 x_3^2.$$

Coefficient of $-p^4 = (a_1 + a_2 + a_3)(-a_1 + a_2 + a_3)(\quad)(\quad) = 0$, from (1).

Coefficient of $2p^2 = -\Sigma a_1^2 x_1^2 (a_1^2 - a_2^2 - a_3^2)$

$$= -\Sigma 2a_1^2 a_2 a_3 x_1^2 \quad \because\ a_1^2 = a_2^2 + a_3^2 + 2a_2 a_3$$
$$= -2a_1 a_2 a_3 \Sigma a_1 x_1^2$$
$$= -2a_1 a_2 a_3 \Sigma x_1^2 (x_2 - x_3)$$
$$= 2a_1^2 a_2^2 a_3^2, \text{ as in (2).}$$

Last term $= 16s'\Pi(s' - ax)$, where $2s' = a_1x_1 + a_2x_2 + a_3x_3$;

$$\therefore\ 2s' = \Sigma x_1 (x_2 - x_3) = 0;$$
$$\therefore\ \text{last term} = 0.$$

Thus $16s\Pi(s - ar) = 4p^2 \Pi a^2$;

$$\therefore\ p^2 \Pi a^2 = 4s\Pi(s - ar).$$

423. [B. 1. a.] *Evaluate in terms of the zeros of $ax^2 + bx + c$, or otherwise, the determinant of the n^{th} order,*

$$D_n = \begin{vmatrix} b+d & c+d & d & d & \dots & d \\ a+d & b+d & c+d & d & \dots & d \\ d & a+d & b+d & c+d & \dots & d \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

in which each diagonal is composed of the same elements and all the elements outside the central three diagonals are the same.

Shew also that the determinant is the coefficient of x^n in the expansion of

$$\frac{1}{(1 - bx + acx^2)} + \frac{dx(1 - acx^2)}{(1 + ax)(1 + cx)(1 - bx + acx^2)^2}.$$

F. S. Macaulay.

Solution by PROPOSER.

Let a, β be the roots of the equation $ax^2+bx+c=0$; add up the elements of D_n by columns, giving for the first row

$$e_n-c,\ e_n,\ e_n,\dots e_n,\ e_n-a,\quad \text{where } e_n=a+b+c+nd.$$

Multiply the rows by $1, a, \dots a^{n-1}$, and add, giving for the first row

$$e'_n-\frac{c}{a},\ e'_n,\ e'_n,\dots e'_n,\ e'_n-aa^n,\quad \text{where } e'_n=d(1+a+\dots+a^{n-1}).$$

Similarly we can obtain for the first row

$$e''_n-\frac{c}{\beta},\ e''_n,\ e''_n,\dots e''_n,\ e''_n-a\beta^n,\quad \text{where } e''_n=d(1+\beta+\dots+\beta^{n-1}).$$

Choose multipliers λ, μ, ν for these three rows, such that

$$\lambda+\mu a^n+\nu\beta^n=0,$$
$$\lambda e_n+\mu e'_n+\nu e''_n=0.$$

We thus make all the elements of the first row zero, with the exception of the first element, which becomes $-c\left(\lambda+\frac{\mu}{a}+\frac{\nu}{\beta}\right)$; while the determinant D_n becomes multiplied by $\lambda+\mu+\nu$.

Hence $$(\lambda+\mu+\nu)D_n=-c\left(\lambda+\frac{\mu}{a}+\frac{\nu}{\beta}\right)D_{n-1};$$

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ 1 & a^n & \beta^n \\ e_n & e'_n & e''_n \end{vmatrix} D_n=-c\begin{vmatrix} 1 & \frac{1}{a} & \frac{1}{\beta} \\ 1 & a_n & \beta^n \\ e_n & e'_n & e''_n \end{vmatrix} D_{n-1}=-a\begin{vmatrix} 1 & 1 & 1 \\ 1 & a^{n+1} & \beta^{n+1} \\ e_n & ae'_n & \beta e''_n \end{vmatrix} D_{n-1}$$

$$=-a\begin{vmatrix} 1 & 1 & 1 \\ 1 & a^{n+1} & \beta^{n+1} \\ e_n+d & ae'_n+d & \beta e''_n+d \end{vmatrix} D_{n-1}=-a\begin{vmatrix} 1 & 1 & 1 \\ 1 & a^{n+1} & \beta^{n+1} \\ e_{n+1} & e'_{n+1} & e''_{n+1} \end{vmatrix} D_{n-1}.$$

Hence

$$\frac{D_n}{(-a)^{n+1}\begin{vmatrix} 1 & 1 & 1 \\ 1 & a^{n+1} & \beta^{n+1} \\ e_{n+1} & e'_{n+1} & e''_{n+1} \end{vmatrix}}=\frac{D_2}{(-a)^3\begin{vmatrix} 1 & 1 & 1 \\ 1 & a^3 & \beta^3 \\ e_3 & e'_3 & e''_3 \end{vmatrix}}=\frac{1}{a(a+b+c)(a-\beta)};$$

$$\therefore\ D_n=\frac{(-)^{n+1}a^n}{(a+b+c)(a-\beta)}\begin{vmatrix} 1 & 1 & 1 \\ 1 & a^{n+1} & \beta^{n+1} \\ e_{n+1} & e'_{n+1} & e''_{n+1} \end{vmatrix}.$$

429. [R. 4. a.] *Three equal smooth spheres of radius r rest in a hollow hemisphere of radius R with their centres in the same horizontal plane; a cone whose weight is equal to the weight of a sphere and whose semivertical angle is a, is inserted symmetrically between the spheres with its vertex downwards; prove that the spheres will separate if*

$$\cot a>\frac{8r}{\sqrt{3R^2-6Rr-r^2}}.$$

Pemb. (c), 1893.

Solution by W. F. BEARD.

Let R be the pressure between each sphere and the bowl.
" R_1 " " " " and the cone.
" R_2 " " " each pair of spheres.

Let O_1, O_2, O_3 be the centres of the three spheres; O the point in which the axis of the cone meets the plane $O_1O_2O_3$.

Then $\qquad O_1O_2O_3$ is an equilateral $\triangle$ with O as its c.g.

Hence $$O_1O=\frac{2r}{\sqrt{3}}.$$

Also, if H is the centre of the hemisphere $O_1H=R-r$.

Let $O\hat{O_1}H=\beta$. Then $\cos\beta=\dfrac{2r}{(R-r)\sqrt{3}}$; $\quad\therefore\ \sin\beta=\dfrac{\sqrt{3R^2-6Rr-r^2}}{(R-r)\sqrt{3}}$. ....(i)

The resultant of the two forces R_2 acting on the sphere centre O_1

$$=2R_2\cos 30^\circ=R_2\sqrt{3}\text{ along }OO_1.$$

Also the force R on this sphere acts along O_1H.

Plainly also the force R_1 acts in the plane OO_1H through O_1, making an $\angle\,\alpha$ with the horizon.

Let W be the weight of each sphere and of the cone.

Consider the equilibrium of the cone.

Resolving vertically, $\quad 3R_1\sin\alpha=W$..(ii)

Consider the equilibrium of the sphere, centre O_1.

Resolving vertically, $\qquad R\sin\beta=W+R_1\sin\alpha=\dfrac{4W}{3}\quad$ from (ii)..........(iii)

Resolving horizontally, $\quad R\cos\beta=R_2\sqrt{3}+R_1\cos\alpha$.

The spheres will separate if R_2 becomes negative or zero,

$$\text{i.e. if } R_1\cos\alpha>R\cos\beta,$$

$$\text{if } \frac{W}{3}\cot\alpha>\frac{4W}{3}\cot\beta,\text{ from (ii) and (iii).}$$

$$\cot\alpha>4\cot\beta$$

$$>\frac{8r}{\sqrt{3R^2-6Rr-r^2}},\text{ from (i).}$$

442. [L^1. 1. b.] [*Mechanical Construction of a Hyperbola.*] *A triangle slides with its base on a fixed straight line: prove that the line joining a fixed point to a carried point cuts a side in a point whose locus is a hyperbola; and find the asymptotes.* R. W. H. T. HUDSON.

Solution by A. W. POOLE.

We may suppose the fixed point on the fixed straight line. Let the $\triangle ABC$ slide along the axis of x. Let O be the fixed point and L the carried point.

Let the coordinates of L be $\qquad l+b,\ a,$

and the coordinates of A be $\qquad l+d,\ c,$

where $l=OB$ and is variable.

Equation of LO is $\dfrac{x}{l+b}=\dfrac{y}{a}$; $\quad$ equation of AB is $\dfrac{x-l}{d}=\dfrac{y}{c}$;

$\therefore$ eliminating l we have $$x+b=\frac{ax}{y}+\frac{dy}{c};$$

or $$dy^2-cxy+acx-bcy=0,$$

or $$(y-a)(dy-cx+\overline{ad-bc})+a\,\overline{ad-bc})=0;$$

an hyperbola passing through O and whose asymptotes are

$$y=a,\quad d(y+a)=c(x+b);$$

$y=a$ passes through L and is parallel to OX; $d(y+a)=c(x+b)$ is parallel to AB and passes through $x=-b$, $y=-a$.

481. [L¹. 10. d.] *PQ is a chord of a parabola meeting the axis in a fixed point H so that $AH=8AS$, and $P'Q'$ its projection on the tangent at the vertex: shew that the locus of the intersection of PQ'. $P'Q$ is a circle.* E. N. BARISIEN.

Solution by J. M. CHILD.

If the coordinates of P, Q be $(ah^2, 2ah)$, $(ak^2, 2ak)$, the coordinates of point of intersection of $P'Q$, PQ' are

$$x=\frac{ah^2k^2}{h^2+k^2},$$

$$y=\frac{2ahk(h+k)}{h^2+k^2};$$

whilst condition that PQ passes through $(0, 8a)$ is

$$hk+8=0.$$

Eliminating h, k we have

$$x^2+y^2-4ax=0,$$

a circle on AK as diameter, where K is the middle point of AH.

483. [L¹. 4. c.] *The tangents to an ellipse at Q and R intersect at a point P on a coaxial ellipse whose axes are respectively n times those of the former curve. Find the locus of the centroid of the triangle PQR.* A. F. VAN DER HEYDEN.

Solution by W. H. SALMON.

Let CP meet the original ellipse in p and QR in V. Then if G be the centroid of PQR,

$$VG=\tfrac{1}{3}VP \text{ and } CP=n\,.\,Cp;$$

$$\therefore\ CG-CV=\tfrac{1}{3}(CP-CV)$$

and

$$CG=\tfrac{1}{3}(CP+2CV)=\tfrac{1}{3}\left(CP+\frac{2Cp^2}{CP}\right)$$

$$=\tfrac{1}{3}\left(n+\frac{2}{n}\right)Cp;$$

$\therefore$ locus of G is a coaxial ellipse, whose axes are $\tfrac{1}{3}\left(n+\frac{2}{n}\right)$ times those of the original ellipse.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) For Sale.

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I.-IX. Edited and Published by E. HENDRICKS, M.A., Des Moines, Iowa, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by ARTEMAS MARTIN, M.A. (Erie, Pa.)]

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. RUNKLE, A.M.

Proceedings of the London Mathematical Society. First series, complete. Vols. 1-35. Bound in 27 vols. Half calf. £25.

Cayley's Mathematical Works. Complete, equal to new, £10. Apply, Professor of Mathematics, University College, Bangor.

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The Messenger of Mathematics. Vols. 2, 15-20, 24, 25.

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Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols.) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

(3) Dr. Muir, The Education Office, Cape Town, will give Vol. 109, *Crelle's Journal*, to any member of the Mathematical Association whose set is without it.

BOOKS, ETC., RECEIVED.

Les Mécanismes. By H. Leblanc. Pp. 432. 1904. (Garnier.)

Leçons de Trigonométrie rectiligne. By C. Bourlet. Pp. xii, 322. 6 frs. 1904. (Armand Colin.)

Leçons sur l'Intégration et la Recherche des Fonctions primitives. By H. Lebesgue. Pp. viii., 136. 3 fr. 50. 1904. (Gauthier-Villars.)

Leçons de Mécanique. By E. Combette and J. Girod. Pp. 321. 3 fr. 50. 1904. (Alcan.)

The Message of Non-Euclidean Geometry. By G. B. Halsted. (Science, March 11, 1904.)

On Generalized Space Differentiation of the Second Order. By B. O. Pierce. Pp. 377-386. (Proceedings of Amer. Acad. of Arts and Sciences.) Feb. 1904.

On the Transfinite Cardinal Numbers of Number Classes in General. By P. E. Jourdain. Pp. 294-303. (Phil. Mag., Mar. 1904.)

The Message of Non-Euclidean Geometry. By G. B. Halsted. Pp. 401-413. (Science, Mar. 11, 1904.)

Elementary and Intermediate Algebra. By J. Lightfoot. Pp. viii, 422; xlii. 2nd Edition. 1904. 4s. 6d. (Ralph, Holland & Co.)

Bibliography of Quaternions and Allied Systems of Mathematics. By A. M'Acfarlane. Pp. 86. N.p. 1904. (Dublin Univ. Press.)

Obras sobre Mathematica. By F. Gomes Teixeira. Published "por ordem do Governo Portuguès." Vol. I. Pp. viii, 402. 1904. (University of Coimbra.)

Notions de Mathématiques. By Jules Tannery. *Notions Historiques.* By Paul Tannery. Pp. x, 352. 1904. (Delagrave.)

History of European Thought in the Nineteenth Century. By J. T. Merz. Vol. II. [Chaps. VI.-XII. On Views of Nature:—Kinetic or mechanical, physical, morphological, genetic, vitalistic, psycho-physical, statistical. Chap. XIII. On the Development of Mathematical Thought during the 19th Century.] Pp. xiii, 807. 15s. net. 1903. (Blackwood.)

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